# Unitary representations for the Schrödinger-Virasoro Lie algebra

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**Abstract**

In this paper, conjugate-linear anti-involutions and unitary Harish-Chandra modules over the Schrödinger-Virasoro algebra are studied. It is proved that there are only two classes conjugate-linear anti-involutions over the Schrödinger-Virasoro algebra. The main result of this paper is that a unitary Harish-Chandra module over the Schrödinger-Virasoro algebra is simply a unitary Harish-Chandra module over the Virasoro algebra.

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# 1 Introduction

The Schrödinger-Virasoro algebra $\mathfrak{sv}$ is defined to be a Lie algebra with $\mathbb{C}$-basis $\{L_n, M_n, Y_{n+\frac{1}{2}}, c \mid n \in \mathbb{Z}\}$ subject to the following Lie brackets:

$$\begin{aligned}
&[L_m, L_n] = (n-m)L_{n+m} + \delta_{m+n,0}\frac{m^3-m}{12}c,\\
&[L_m, M_n] = nM_{n+m},\\
&[L_m, Y_{n+\frac{1}{2}}] = (n+\frac{1-m}{2})Y_{m+n+\frac{1}{2}},\\
&[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n-m)M_{m+n+1},\\
&[M_m, M_n] = 0 = [M_m, Y_{n+\frac{1}{2}}] = [\mathfrak{sv}, c].
\end{aligned}$$

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It was introduced by M. Henkel in Ref. [7] by looking at the invariance of the free Schrödinger equation. Due to its important roles in mathematics and statistical physics, it has been studied extensively by many authors. In Refs. [17, 19, 20], the twisted Schrödinger-Virasoro algebra, $\varepsilon$-deformation Schrödinger-Virasoro algebra, the generalized Schrödinger-Virasoro algebra and the extended Schrödinger-Virasoro algebras are introduced. These Lie algebras are all natural deformations of the Schrödinger-Virasoro algebra $\mathfrak{sv}$. In Refs [6, 17, 19, 21], the derivations, the 2-cocycles, the central extensions and the automorphisms for these algebras have been studied.

It is well known that the Virasoro Lie algebra is an important Lie algebra, whose representation theory plays a crucial role in many areas of mathematics and physics. Many representations, such as the Harish-Chandra modules, the Verma modules and the Whittaker modules, of it have been well studied (cf. Refs. [1, 3, 5, 8, 11, 13-16, 18]). The Virasoro Lie algebra is a subalgebra of the Schrödinger-Virasoro algebra, it is natural to consider those representations for the Schrödinger-Virasoro algebra. In Refs [12, 19, 20, 22], the Harish-Chandra modules, the Verma modules, the vertex algebra representations and the Whittaker modules over the Schrödinger-Virasoro algebra are studied.

In Ref. [1, 2-5], the nontrivial unitary irreducible unitary modules are classified and the unitary highest weight modules of the Virasoro algebra are well studied. Motivated by these work, we consider the unitary modules over the Schrödinger-Virasoro algebra in this paper. The paper is organized as follows. In section 2, we prove that there are only two classes conjugate-linear anti-involutions over the Schrödinger-Virasoro algebra $\mathfrak{sv}$. In section 3, we prove that a unitary weight module over the Schrödinger-Virasoro algebra is simply a unitary weight module over the Virasoro algebra. Then the unitary weight modules over $\mathfrak{sv}$ are classified since that the ones over the Virasoro algebra are classified.

Throughout this paper we make a convention that the weight modules over the Schrödinger-Virasoro algebra and Virasoro algebra are all with finite dimensional weight spaces, i.e., the Harish-Chandra modules. The symbols $\mathbb{C}, \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ represent for the complex field, the set of nonnegative integers, the set of integers, the set of positive integers and the set of negative integers respectively.

# 2 Conjugate-linear anti-involution of $\mathfrak{sv}$

It is easy to see the following facts about $\mathfrak{sv}$ :

(i) $\mathfrak{C} := \mathbb{C}M_0 \oplus \mathbb{C}c$ is the center of $\mathfrak{sv}$.

(ii) If $x \in \mathfrak{sv}$ acts semisimply on $\mathfrak{sv}$ by the adjoint action, then $x \in \mathfrak{h}$, where $\mathfrak{h} := \mathrm{span}_{\mathbb{C}}\{L_0, M_0, c\}$ is the unique Cartan subalgebra of $\mathfrak{sv}$.

(iii) $\mathfrak{sv}$ has a weight space decomposition according to the Cartan subalgebra $\mathfrak{h}$ :

$$\mathfrak{sv} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{sv}_n \oplus \bigoplus_{n\in\mathbb{Z}} \mathfrak{sv}_{\frac{1}{2}+n},$$

where $\mathfrak{sv}_n = \mathrm{span}_{\mathbb{C}}\{L_n, M_n\}$, $\mathfrak{sv}_{\frac{1}{2}+n} = \mathrm{span}_{\mathbb{C}}\{Y_{\frac{1}{2}+n}\}$, $n \in \mathbb{Z}$.

If we denote $Vir = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n \oplus \mathbb{C}c$, $M = \oplus_{n\in\mathbb{Z}}\mathbb{C}M_n, Y = \oplus_{n\in\mathbb{Z}}\mathbb{C}Y_{\frac{1}{2}+n}$. Then we have the following lemma:

**Lemma 2.1.** *$M \oplus Y \oplus \mathbb{C}c$ is the unique maximal ideal of $\mathfrak{sv}$.*

**Proof.** The proof is similar as that for Lemma 2.2 in Ref. [19]. □

**Definition 2.2.** Let $\mathfrak{g}$ be a Lie algebra and $\theta$ be a conjugate-linear anti-involution of $\mathfrak{g}$, i.e. $\theta$ is a map $\mathfrak{g} \to \mathfrak{g}$ such that

$$\theta(x+y) = \theta(x) + \theta(y),\ \theta(\alpha x) = \bar{\alpha}\theta(x),$$

$$\theta([x,y]) = [\theta(y), \theta(x)],\ \theta^2 = \mathrm{id}$$

for all $x, y \in \mathfrak{g},\ \alpha \in \mathbb{C}$, where id is the identity map of $\mathfrak{g}$. A module $V$ of $\mathfrak{g}$ is called unitary if there is a positive definite Hermitian form $\langle\ ,\ \rangle$ on $V$ such that

$$\langle xu, v\rangle = \langle u, \theta(x)v\rangle$$

for all $u, v \in V, x \in \mathfrak{g}$.

**Lemma 2.3.** (Proposition 3.2 in Ref. [1]) *Any conjugate-linear anti-involution of $Vir$ is one of the following types:*

*(i) $\theta_\alpha^+(L_n) = \alpha^n L_{-n},\ \theta_\alpha^+(c) = c$, for some $\alpha \in \mathbb{R}^\times$, the set of nonzero real number.*

*(ii) $\theta_\alpha^-(L_n) = -\alpha^n L_n,\ \theta_\alpha^-(c) = -c$, for some $\alpha \in S^1$, the set of complex number of modulus one.*

**Lemma 2.4.** *Let $\theta$ be an arbitrary conjugate-linear anti-involution of $\mathfrak{sv}$. Then*

*(i) $\theta(M \oplus Y) = M \oplus Y$.*

*(ii) $\theta(\mathfrak{h}) = \mathfrak{h}$.*

*(iii) $\theta(c) = \lambda c + \lambda' M_0$, $\theta(M_0) = \mu M_0$, where $\lambda, \mu \in S^1, \lambda' \in \mathbb{C}$.*

**Proof.** (i), $\forall x \in \mathfrak{sv}, y \in M \oplus Y$, the identity $[x, \theta(y)] = \theta([y, \theta(x)])$ means that $\theta(M \oplus Y)$ is an ideal of $\mathfrak{sv}$. Thus $\theta(M \oplus Y) \subseteq M \oplus Y \oplus \mathbb{C}c$ by Lemma 2.1. Assume that

$$\theta(Y_{\frac{1}{2}+n}) = a_n x + \beta_n c,$$

where $a_n, \beta_n \in \mathbb{C}, x \in M \oplus Y$. Then by $[L_0, Y_{\frac{1}{2}+n}] = (\frac{1}{2}+n)Y_{\frac{1}{2}+n}$, we see that $\beta_n = 0$. Moreover, $\theta(M_n) \subseteq M \oplus Y$ since $M = [Y, Y]$.

For (ii), $\forall x \in \mathfrak{sv}$, $[x, \theta(M_0)] = \theta([M_0, \theta(x)]) = 0$. So $\theta(M_0) \in \mathfrak{C}$. Similarly, $\theta(c) \in \mathfrak{C}$. The identities

$$[\theta(L_0), \theta(L_n)] = -n\theta(L_n),\ [\theta(L_0), \theta(M_n)] = -n\theta(M_n),$$

and

$$[\theta(L_0), \theta(Y_{\frac{1}{2}+n})] = -(\frac{1}{2}+n)\theta(Y_{\frac{1}{2}+n})$$

imply that $\theta(L_0)$ acts semisimply on $\mathfrak{sv}$. Thus $\theta(L_0) \in \mathfrak{h}$.

For (iii), note that $\mathfrak{C}$ is the center of $\mathfrak{sv}$, we have $\theta(\mathfrak{C}) = \mathfrak{C}$, so we can assume

$$\theta(c) = \lambda c + \lambda^{'} M_0.$$

Since $\theta(M_0) \in (M \oplus Y) \cap \mathfrak{C}$, we can assume $\theta(M_0) = \mu M_0$. So $M_0 = \theta^2(M_0) = \mu\bar{\mu}M_0$, thus $\mu \in S^1$. Similarly, we have $\lambda \in S^1$. □

**Proposition 2.5.** Any conjugate-linear anti-involution of $\mathfrak{sv}$ is one of the following types:

$$\begin{aligned}
\text{(i)}: \ \ &\theta^+_{\alpha,\beta,\mu}(L_n) = \alpha^n L_{-n} + (\frac{n+1}{2}\alpha^{n-1}\beta + \frac{n-1}{2}\alpha^{n-1}\mu\overline{\beta})M_{-n},\\
&\theta^+_{\alpha,\beta,\mu}(c) = c,\\
&\theta^+_{\alpha,\beta,\mu}(M_n) = \mu\alpha^n M_{-n},\\
&\theta^+_{\alpha,\beta,\mu}(Y_{\frac{1}{2}+n}) = \mu^{\frac{1}{2}}\alpha^{\frac{1}{2}+n}Y_{-\frac{1}{2}-n}
\end{aligned}$$

for some $\alpha \in \mathbb{R}^\times, \mu \in S^1, \beta \in \mathbb{C}$.

$$\begin{aligned}
\text{(ii)}: \ \ &\theta^-_{\alpha,r_1,r_2,\mu}(L_n) = -\alpha^n L_n + (\frac{n+1}{2}\alpha^{n+1}\mu r_1 - \frac{n-1}{2}\alpha^{n-1}\mu r_2)M_n,\\
&\theta^-_{\alpha,r_1,r_2,\mu}(c) = -c,\\
&\theta^-_{\alpha,r_1,r_2,\mu}(M_n) = \mu\alpha^n M_n,\\
&\theta^-_{\alpha,r_1,r_2,\mu}(Y_{\frac{1}{2}+n}) = (-\mu)^{\frac{1}{2}}\alpha^{\frac{1}{2}+n}Y_{\frac{1}{2}+n}
\end{aligned}$$

for some $\alpha, \mu \in S^1, r_1, r_2 \in \mathbb{R}$.

**Proof.** Let $\theta$ be any conjugate-linear anti-involution of $\mathfrak{sv}$. By Lemma 2.4 (i), we have the induced conjugate-linear anti-involution of $\mathfrak{sv}/(M \oplus Y) \simeq Vir$ :

$$\bar{\theta} : \mathfrak{sv}/(M \oplus Y) \to \mathfrak{sv}/(M \oplus Y).$$

Thus by Lemma 2.3 we see that $\bar{\theta}$ is one of the following types:

(a) $\bar{\theta}^+_\alpha(\bar{L_n}) = \alpha^n \bar{L_{-n}}$, $\bar{\theta}^+_\alpha(\bar{c}) = \bar{c}$, for some $\alpha \in R^\times$.

(b) $\bar{\theta}^-_\alpha(\bar{L_n}) = -\alpha^n \bar{L_n}$, $\bar{\theta}^-_\alpha(\bar{c}) = -\bar{c}$, for some $\alpha \in S^1$.

If $\bar{\theta}$ is of type (a), we can assume

$$\theta(L_n) = \alpha^n L_{-n} + \sum_i \beta_{n,i}M_i + \sum_j \gamma_{n,j}Y_{\frac{1}{2}+j}, \tag{2.1}$$

where $\beta_{n,i}, \gamma_{n,j}, a_n \in \mathbb{C}$. By (2.1) and Lemma 2.4 (ii), we have

$$\theta(L_0) = L_0 + \beta_{0,0}M_0 + a_0 c.$$

Then by $[\theta(L_{-1}), \theta(L_1)] = -2\theta(L_0)$, we deduce that $a_0 = 0$. Thus

$$\theta(L_0) = L_0 + \beta_{0,0}M_0. \tag{2.2}$$

By (2.1), (2.2) and the identity $[\theta(L_n), \theta(L_0)] = n\theta(L_n)$, it can be deduced easily that $\beta_{n,i} = 0$ unless $i = -n$, $\gamma_{n,j} = 0$ for all $j \in \mathbb{Z}$, i.e.,

$$\theta(L_n) = \alpha^n L_{-n} + \beta_{n,-n} M_{-n}. \tag{2.3}$$

By (2.3) and the identity $[\theta(L_n), \theta(L_m)] = (n-m)\theta(L_{m+n}) + \delta_{m+n,0}\frac{n-n^3}{12}\theta(c)$, we can get

$$((n-m)\beta_{m+n,-(m+n)} - n\beta_{n,-n}\alpha^m + m\alpha^n\beta_{m,-m})M_{-(m+n)}$$

$$= \delta_{m+n,0}\frac{n-n^3}{12}((1-\lambda)c - \lambda^{'} M_0) \tag{2.4}$$

Let $m = -n \neq -1, 0, 1$ in (2.4), we see that

$$\lambda = 1. \tag{2.5}$$

Let $m = -n = 1$ in (2.4), we have

$$\beta_{0,0} = \frac{\alpha^{-1}}{2}\beta_{1,-1} + \frac{\alpha}{2}\beta_{-1,1}. \tag{2.6}$$

Let $m = -n = 2$ in (2.4), we have

$$\lambda^{'} = 8\beta_{0,0} - 4\alpha^2\beta_{-2,2} - 4\alpha^{-2}\beta_{2,-2}. \tag{2.7}$$

Let $m = 2, n = -1$ and $m = -2, n = 1$ in (2.4) respectively , we have

$$\beta_{2,-2} = \frac{3\alpha}{2}\beta_{1,-1} - \frac{\alpha^3}{2}\beta_{-1,1}, \quad \beta_{-2,2} = \frac{3\alpha^{-1}}{2}\beta_{-1,1} - \frac{\alpha^{-3}}{2}\beta_{1,-1}. \tag{2.8}$$

By (2.6)-(2.8), we have

$$\lambda^{'} = 0. \tag{2.9}$$

By (2.4), (2.5), (2.9) and Lemma 2.4 (iii), we have

$$\theta(c) = c, \tag{2.10}$$

and

$$(n-m)\beta_{m+n,-(m+n)} = n\beta_{n,-n}\alpha^m - m\alpha^n\beta_{m,-m}. \tag{2.11}$$

Let $n = 1$ in (2.11), we have

$$(1-m)\beta_{m+1,-(m+1)} + m\alpha\beta_{m,-m} - \alpha^m\beta_{1,-1} = 0.$$

Then using induction, we can prove that, for $m \geq 1$,

$$\beta_{m,-m} = -(m-2)\alpha^{m-1}\beta_{1,-1} + (m-1)\alpha^{m-2}\beta_{2,-2}. \tag{2.12}$$

Then (2.8) and (2.12) give us that

$$\beta_{m,-m} = \frac{m+1}{2}\alpha^{m-1}\beta_{1,-1} - \frac{m-1}{2}\alpha^{m+1}\beta_{-1,1},\ (\forall m \in \mathbb{Z}_+). \tag{2.13}$$

Let $n = -1$ in (2.11) and by a similar argument as above, we can prove that

$$\beta_{m,-m} = \frac{m+1}{2}\alpha^{m-1}\beta_{1,-1} - \frac{m-1}{2}\alpha^{m+1}\beta_{-1,1}, (\forall m \in \mathbb{Z}_-). \tag{2.14}$$

Then by (2.6), (2.13) and (2.14), we see that

$$\beta_{m,-m} = \frac{m+1}{2}\alpha^{m-1}\beta_{1,-1} - \frac{m-1}{2}\alpha^{m+1}\beta_{-1,1}, (\forall m \in \mathbb{Z}). \tag{2.15}$$

Now by (2.3) and (2.15), we have

$$\theta(L_n) = \alpha^n L_{-n} + (\frac{n+1}{2}\alpha^{n-1}\beta_{1,-1} - \frac{n-1}{2}\alpha^{n+1}\beta_{-1,1})M_{-n}. \tag{2.16}$$

By Lemma 2.4 (i), we can assume that

$$\theta(M_m) = \sum_i \zeta_{m,i}M_i + \sum_j \xi_{m,j}Y_{\frac{1}{2}+j}, \tag{2.17}$$

$$\theta(Y_{\frac{1}{2}+n}) = \sum_i \lambda_{n,i}M_i + \sum_j \mu_{n,j}Y_{\frac{1}{2}+j}, \tag{2.18}$$

where $\zeta_{m,i},\ \xi_{m,j},\ \lambda_{n,i},\ \mu_{n,j} \in \mathbb{C}$. By (2.16), (2.17), Lemma 2.4 (iii) and the identity

$$[\theta(M_{-n}), \theta(L_n)] = -n\theta(M_0)$$

we get that $\zeta_{-n,i} = 0$ for all $i \neq n$, $\alpha^n\zeta_{-n,n} = \mu$, $\xi_{-n,j} = 0$ for all $j \in \mathbb{Z}$. Thus

$$\theta(M_n) = \alpha^n \mu M_{-n}. \tag{2.19}$$

By (2.3) and (2.19) we have

$$L_1 = \theta^2(L_1) = L_1 + (\alpha\beta_{-1,1} + \alpha^{-1}\mu\overline{\beta_{1,-1}})M_1.$$

Thus

$$\beta_{-1,1} = -\alpha^{-2}\mu\overline{\beta_{1,-1}} \tag{2.20}$$

By (2.16) and (2.20), we have

$$\theta(L_n) = \alpha^n L_{-n} + (\frac{n+1}{2}\alpha^{n-1}\beta + \frac{n-1}{2}\alpha^{n-1}\mu\overline{\beta})M_{-n}, \tag{2.21}$$

where $\beta = \beta_{1,-1}$.

By (2.16), (2.18) and the identity $[\theta(Y_{\frac{1}{2}+m}),\theta(L_0)] = (\frac{1}{2}+m)\theta(Y_{\frac{1}{2}+m})$, we get that $\lambda_{m,i} = 0$ for all $i \in \mathbb{Z}$, $\mu_{m,j} = 0$ unless $j = -(m+1)$. Thus

$$\theta(Y_{\frac{1}{2}+m}) = a_{\frac{1}{2}+m} Y_{-\frac{1}{2}-m}, \tag{2.22}$$

where $a_{\frac{1}{2}+m} = \mu_{m,-m-1}$. If $n \neq 1$, by (2.16), (2.22) and the identity $[\theta(Y_{\frac{1}{2}}),\theta(L_n)] = (\frac{1-n}{2})\theta(Y_{\frac{1}{2}+n})$, we have

$$\frac{1-n}{2} a_{\frac{1}{2}} \alpha^n Y_{-\frac{1}{2}-n} = \frac{1-n}{2} a_{\frac{1}{2}+n} Y_{-\frac{1}{2}-n}.$$

Then $a_{\frac{1}{2}+n} = \alpha^n a_{\frac{1}{2}}, (n \neq 1)$. By $[\theta(Y_{\frac{3}{2}}),\theta(L_{-2})] = \theta([L_{-2}, Y_{\frac{3}{2}}])$, we can easily get that $a_{\frac{1}{2}+1} = \alpha a_{\frac{1}{2}}$. So we have

$$a_{\frac{1}{2}+n} = \alpha^n a_{\frac{1}{2}}, \forall n \in \mathbb{Z}. \tag{2.23}$$

By (2.19), (2.22) and the identity $[\theta(Y_{\frac{1}{2}+m}),\theta(Y_{-\frac{1}{2}-m})] = \theta([Y_{-\frac{1}{2}-m}, Y_{\frac{1}{2}+m}])$, we have

$$a_{\frac{1}{2}+m} a_{-\frac{1}{2}-m} (2m+1) M_0 = \mu(2m+1) M_0.$$

Thus

$$a_{\frac{1}{2}+m} a_{-\frac{1}{2}-m} = \mu. \tag{2.24}$$

By (2.23) and (2.24), we see that

$$a_{\frac{1}{2}} = \sqrt{\mu\alpha}. \tag{2.25}$$

By (2.22), (2.23) and (2.25), we have

$$\theta(Y_{\frac{1}{2}+m}) = \sqrt{\mu}\alpha^{\frac{1}{2}+m} Y_{-\frac{1}{2}-m}. \tag{2.26}$$

Now (i) follows from (2.10), (2.19), (2.21) and (2.26).

If $\bar{\theta}$ is of type (b), by a similar discussion in the way of (2.1)-(2.16), we can prove that

$$\theta(L_n) = -\alpha^n L_n + (\frac{n+1}{2}\alpha^{n-1}\beta_1 - \frac{n-1}{2}\alpha^{n+1}\beta_{-1}) M_n. \tag{2.27}$$

$$\theta(c) = -c, \tag{2.28}$$

where $\alpha \in S^1, \beta_1, \beta_{-1} \in \mathbb{C}$. By a similar discussion in the way of (2.17)-(2.19) and (2.22)-(2.26), we have

$$\theta(M_n) = \mu\alpha^n M_n, \tag{2.29}$$

$$\theta(Y_{\frac{1}{2}+n}) = (-\mu)^{\frac{1}{2}} \alpha^{\frac{1}{2}+n} Y_{\frac{1}{2}+n}, \tag{2.30}$$

where $\mu \in S^1$. By (2.27), (2.29) and the identities $\theta^2(L_1) = L_1$ and $\theta^2(L_{-1}) = L_{-1}$, we see that

$$\overline{\alpha}\beta_1 = \overline{\beta_1}\alpha\mu, \ \alpha\beta_{-1} = \overline{\beta_{-1}}\overline{\alpha}\mu.$$

If we set $\alpha = e^{i\sigma}, \mu = e^{i\tau}, \beta_1 = |\beta_1|e^{ix}$, then by $\overline{\alpha}\beta_1 = \overline{\beta_1}\alpha\mu$, we see that

$$\beta_1 = |\beta_1|e^{i(2\sigma+\tau)} \text{ or } -|\beta_1|e^{i(2\sigma+\tau)}.$$

Similarly,

$$\beta_{-1} = |\beta_{-1}|e^{i(-2\sigma+\tau)} \text{ or } -|\beta_{-1}|e^{i(-2\sigma+\tau)}.$$

We set $r_1, r_2 \in \mathbb{R}$ such that $|r_1| = |\beta_1|$ and $|r_2| = |\beta_{-1}|$. Then

$$\beta_1 = r_1\alpha^2\mu,\ \beta_{-1} = r_2\overline{\alpha}^2\mu. \tag{2.31}$$

Thus (ii) follows from (2.27)-(2.31). □

The following Lemma is crucial for the proof of Proposition 3.4.

**Lemma 2.6.** Let $\theta$ be a conjugate-linear anti-involution of the Schrödinger-Virasoro algebra $\mathfrak{sv}$.

(i) If $\theta = \theta^+_{\alpha,\beta,\mu}$, we denote by $Vir'$ the subalgebra of $\mathfrak{sv}$ generated by

$$\{c, L_n' := L_n - \frac{n-1}{2}\alpha^{-1}\beta M_n \mid n \in \mathbb{Z}\}.$$

Then $Vir' \simeq Vir$ and $\theta^+_{\alpha,\beta,\mu}(L_n') = \alpha^n L'_{-n}, \theta^+_{\alpha,\beta,\mu}(c) = c$.

(ii) If $\theta = \theta^-_{\alpha,r_1,r_2,\mu}$, we denote by $Vir'$ the subalgebra of $\mathfrak{sv}$ generated by

$$\{c, L_n' := L_n + x_n M_n \mid n \in \mathbb{Z}\},$$

where $x_n \in \mathbb{C}$ satisfying $\overline{x_n}\mu^{\frac{1}{2}} + x_n\mu^{-\frac{1}{2}} = \frac{n-1}{2}r_2 - \frac{n+1}{2}r_1$. Then $Vir' \simeq Vir$ and $\theta^-_{\alpha,r_1,r_2,\mu}(L_n') = -\alpha^n L_n', \theta^-_{\alpha,r_1,r_2,\mu}(c) = -c$.

**Proof.** It can be checked directly, we omit the details. □

**Lemma 2.7.** (Proposition 3.4 in Ref. [1]) Let $V$ be a nontrivial irreducible weight $Vir$-module.

(i) If $V$ is unitary for some conjugate-linear anti-involution $\theta$ of $Vir$, then $\theta = \theta^+_\alpha$ for some $\alpha > 0$.

(ii) If $V$ is unitary for $\theta^+_\alpha$ for some $\alpha > 0$, then $V$ is unitary for $\theta^+_1$.

**Proposition 2.8.** Let $V$ be a nontrivial irreducible weight $\mathfrak{sv}$-module.

(i) If $V$ is unitary for some conjugate-linear anti-involution $\theta$ of $\mathfrak{sv}$, then $\theta = \theta^+_{\alpha,\beta,\mu}$ for some $\alpha > 0$.

(ii) If $V$ is unitary for $\theta^+_{\alpha,\beta,\mu}$ for some $\alpha > 0$, then $V$ is unitary for $\theta^+_{1,\beta,\mu}$.

**Proof.** (i) Suppose $V$ is unitary for some conjugate-linear anti-involution $\theta$ of $\mathfrak{sv}$. By Lemma 2.6, $V$ can be viewed as a unitary $Vir'$-module for the conjugate-linear anti-involution $\theta|_{Vir'}$. Then $V$ is a direct sum of irreducible unitary $Vir'$-modules since any unitary weight $Vir$-module is complete reducible. We claim that $V$ is a

nontrivial $Vir'$-module. Otherwise, for any $0 \neq v \in V, n \in \mathbb{Z} \setminus 0, m \in \mathbb{Z}$, we have

$$
\begin{aligned}
M_0 v &= -\frac{1}{n}(L'_n M_{-n} - M_{-n} L'_n)v = 0, \\
M_n v &= \frac{1}{n}(L'_0 M_n - M_n L'_0)v = 0, \\
Y_{\frac{1}{2}+m} v &= \frac{2}{1+2m}(L'_0 Y_{\frac{1}{2}+m} - Y_{\frac{1}{2}+m} L'_0)v = 0.
\end{aligned}
$$

So $\mathfrak{sv}.V = 0$, a contradiction. Thus there is a nontrivial irreducible unitary $Vir'$-submodule of $V$ for conjugate-linear anti-involution $\theta|_{Vir'}$. By Lemma 2.7, $\theta|_{Vir'} = \theta^+_\alpha$ for some $\alpha > 0$. Then by Proposition 2.5, we have $\theta = \theta^+_{\alpha,,\beta,\mu}$ for some $\mu \in S^1, \beta \in \mathbb{C}$.

(ii) Suppose $V$ is unitary for $\theta^+_{\alpha,,\beta,\mu}$ for some $\alpha \in \mathbb{R}^\times, \mu \in S^1, \beta \in \mathbb{C}$ and $\langle \ . \ \rangle_\alpha$ is the Hermitian form on $V$. We can assume $V$ is generated by a $L'_0$-eigenvector $v_0$ with eigenvalue $a \in \mathbb{C}$ since $V$ is irreducible weight $\mathfrak{sv}$-module. By

$$\langle L'_0 v_0, v_0 \rangle_\alpha = \langle v_0, L'_0 v_0 \rangle_\alpha,$$

we see that $a \in \mathbb{R}$. Then $L'_0$-eigenvalues on $V$ are of the form $a + \frac{n}{2}$, $n \in \mathbb{Z}$. Define a new form $\langle \ , \ \rangle$ on $V$ by

$$\langle v, w \rangle = \alpha^{-\frac{n}{2}} \langle v, w \rangle_\alpha, \forall v, w \in \{v \mid L'_0 v = (a + \frac{n}{2})v\}.$$

It is easy to check that this form makes $V$ unitary with the conjugate-linear anti-involution $\theta^+_{1,\beta,\mu}$. $\square$

# 3 Unitary representations for $\mathfrak{sv}$

In this section, we study the unitary weight modules for $\mathfrak{sv}$. By Prop. 2.8, we see that the conjugate-linear anti-involution is of the form $\theta^+_{\alpha,\beta,\mu}$ for some $\alpha > 0$. For the sake of simplicity, we write $\theta^+_{\alpha,\beta,\mu}$ by $\theta$.

It is known that a unitary weight module over Virasoro algebra is completely reducible. This result also holds for $\mathfrak{sv}$ :

**Lemma 3.1.** If $V$ is a unitary weight module for $\mathfrak{sv}$, then $V$ is completely reducible.

**Proof.** Let $N$ be a submodule. Then $N^+ := \{v \in V | \langle v, N \rangle = 0\}$ is a submodule of $V$ since for any $v \in N^+$, $\langle x.v, N \rangle = \langle v, \theta(x)N \rangle = 0$. It is well known that any submodule of a weight module is a weight module. For any weight $\lambda$ of $V$, denote by $V_\lambda, N_\lambda, N^+_\lambda$ the weight space with weight $\lambda$ of $V, N, N^+$ respectively. It is obvious that $dim(V_\lambda) < \infty$ since $V$ is a Harish-Chandra module, so we can extend an orthogonal basis of $N_\lambda$ as an orthogonal basis of $V_\lambda$, thus we have

$$V_\lambda = N_\lambda \oplus N^+_\lambda,$$

which means $V = N \oplus N^+$. $\square$

**Lemma 3.2.** (Theorem 1.3 (i) in [12]) An irreducible weight module over $\mathfrak{sv}$ is either a highest/lowest weight module or a uniformly bounded one.

It is well known (See Refs. [10] and [11]) that there are three types modules of the intermediate series over $Vir$, denoted respectively by $A_{a,b}, A_\alpha, B_\beta$, they all have basis $\{v_k \mid k \in \mathbb{Z}\}$ such that $c$ acts trivially and

$$\begin{aligned}
&A_{a,b} : L_n v_k = (a + k + nb) v_{n+k};\\
&A_\alpha : L_n v_k = (n + k) v_{n+k} \text{ if } k \neq 0,\ L_n v_0 = n(n + a) v_n;\\
&B_\beta : L_n v_k = k v_{n+k} \text{ if } k \neq -n,\ L_n v_{-n} = -n(n + a) v_0.
\end{aligned}$$

for all $n, k \in \mathbb{Z}$. For the irreducible modules of the intermediate series of type $A_{a,b}$, we have fact that: $A_{a,b}$ and $A_{c,d}$ are isomorphic if and only if $a - c \in \mathbb{Z}$ and $b = d$ or $1 - d$.

**Lemma 3.3.** (Theorem 0.5 in [1]) Let $V$ be an irreducible unitary module of $Vir$ with finite-dimensional weight spaces. Then either $V$ is highest or Lowest weight, or $V$ is isomorphic to $A_{a,b}$ for some $a \in \mathbb{R}, b \in \frac{1}{2} + \sqrt{-1}\mathbb{R}$.

**Proposition 3.4.** A unitary weight module over $\mathfrak{sv}$ is simply a unitary weight module over $Vir$. That is, if $V$ is a unitary weight module over $\mathfrak{sv}$, then $M.V = Y.V = 0$.

**Proof.** Let $V$ be a unitary weight module over $\mathfrak{sv}$ for a conjugate-linear anti-involution $\theta$. By Lemma 2.6 (i), $V$ is also unitary for $Vir'$, thus the well known result for the unitary modules over Virasoro Lie algebra can be used freely. By Lemma 3.1, we may assume that $V$ is irreducible. By Lemma 3.2, it is sufficient to consider the following two cases:

**Case 1.** $V$ is a unitary irreducible highest/lowest weight module. Let $v_\lambda$ be a highest weight vector. For $n \in \mathbb{Z}_+$, we have $\langle M_{-n} v_\lambda, M_{-n} v_\lambda \rangle = \langle v_\lambda, \mu M_{-n} M_n v_\lambda \rangle = 0$, thus $M_{-n} v_\lambda = 0$. Furthermore,

$$\langle L'_{-n} v_\lambda, M_{-n} v_\lambda \rangle = \langle v_\lambda, \alpha^{-n} L'_n M_{-n} v_\lambda \rangle = -n\alpha^{-n} \lambda(M_0) \langle v_\lambda, v_\lambda \rangle = 0.$$

So $M_0 v_\lambda = 0$. Thus

$$M.V = 0.$$

For $n \in \mathbb{N}$, note that $M_0 v_\lambda = 0$, we have

$$\langle Y_{-\frac{1}{2}-n} v_\lambda, Y_{-\frac{1}{2}-n} v_\lambda \rangle = \langle v_\lambda, \mu^{\frac{1}{2}} Y_{\frac{1}{2}+n} Y_{-\frac{1}{2}-n} v_\lambda \rangle = 0.$$

Thus $Y_{-\frac{1}{2}-n} v_\lambda = 0$, which means that

$$Y.V = 0.$$

**Case 2.** $V$ is a unitary irreducible uniformly bounded module. As $Vir'$-module, $V$ is a direct sum of unitary irreducible $Vir'$-submodules, So by Lemma 3.3 we can suppose that

$$V = A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K} \oplus W,$$

where $a_i \in \mathbb{R}, b_i \in \frac{1}{2} + \sqrt{-1}\mathbb{R}$, $W$ is a trivial $Vir'$-module.

**Subcase 2.1.** $W = 0$.

In this subcase

$$V = A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K}.$$

Let $\{v_k \mid k \in \mathbb{Z}\}$ be a basis of $A_{a_1,b_1}$ such that $L_n' v_k = (a_1 + k + nb_1)v_{n+k}$. As an irreducible $\mathfrak{sv}$-module, $V$ is generated by the $L_0'$-eigenvector $v_0$ with eigenvalue $a_1$. Thus $L_0'$-eigenvalue on $V$ are of the form $a_1 + \frac{n}{2}, n \in \mathbb{Z}$. This means that $a_i \in \{a_1 + \frac{n}{2} \mid n \in \mathbb{Z}\}$, $i = 1, \cdots, K$. Recall that $A_{a+n,b} \simeq A_{a,b}$ for any $n \in \mathbb{Z}$. So there exists $0 \le a < \frac{1}{2}$ such that $A_{a_i,b_i}$ are of the form $A_{a,b_i}$ or $A_{\frac{1}{2}+a,b_i}$. i.e.,

$$V = A_{a,b_1} \oplus \cdots \oplus A_{a,b_R} \oplus A_{\frac{1}{2}+a,d_1} \oplus \cdots \oplus A_{\frac{1}{2}+a,d_S},$$

where $0 \le a < \frac{1}{2}, b_i, d_j \in \frac{1}{2} + \sqrt{-1}\mathbb{R}$. Now we choose a basis

$$\{v_{k,l} \mid k \in \mathbb{Z}, 1 \le l \le R\} \cup \{v_{\frac{1}{2}+k,l'} \mid k \in \mathbb{Z}, 1 \le l' \le S\}$$

such that

$$L_m' v_{k,l} = (a + k + mb_l)v_{k+m,l}, \tag{3.1}$$

$$L_m' v_{\frac{1}{2}+k,l'} = (\frac{1}{2} + a + k + md_{l'})v_{\frac{1}{2}+k+m,l'} \tag{3.2}$$

for $m \in \mathbb{Z}, 1 \le l \le R, 1 \le l' \le S$. Suppose

$$Y_{\frac{1}{2}} v_{k,l} = \sum_{l'=1}^{S} \mu_{k,l}^{l'} v_{\frac{1}{2}+k,l'}, \tag{3.3}$$

$$Y_{\frac{1}{2}} v_{\frac{1}{2}+k,l'} = \sum_{l=1}^{R} \lambda_{k,l'}^{l} v_{k+1,l}. \tag{3.4}$$

**Claim.** $Y_{\frac{1}{2}} v_{k,l} = 0 = Y_{\frac{1}{2}} v_{\frac{1}{2}+k,l'}, \ \forall k, l.$

Suppose the claim holds, Then $Y_{\frac{1}{2}}.V = 0$. Note that $Y, M$ can be generated by $Y_{\frac{1}{2}}$ and $Vir'$, we obtain $M.V = 0 = Y.V$, as desired. So it is sufficient to prove the claim.

***proof of the claim.*** By (3.1), (3.3) and the identity $[L_1', Y_{\frac{1}{2}}] = 0$, we can easily deduce that

$$(a + k + b_l)\mu_{k+1,l}^{l'} = (a + \frac{1}{2} + k + d_{l'})\mu_{k,l}^{l'}. \tag{3.5}$$

By (3.1)-(3,3) and the identity $[L'_{-1}, Y_{\frac{1}{2}}] = Y_{-\frac{1}{2}}$, we have

$$Y_{-\frac{1}{2}}v_{k,l} = \sum_{l'=1}^{S}((a+\frac{1}{2}+k-d_{l'})\mu_{k,l}^{l'} - (a+k-b_l)\mu_{k-1,l}^{l'})v_{\frac{1}{2}+(k-1),l'}. \tag{3.6}$$

By (3.1), (3.2), (3.6) and the identity $[L'_1, Y_{-\frac{1}{2}}] = -Y_{\frac{1}{2}}$, we have

$$\begin{aligned}&\sum_{l'=1}^{S}((a+\frac{1}{2}+k-d_{l'})\mu_{k,l}^{l'} - (a+k-b_l)\mu_{k-1,l}^{l'})(a-\frac{1}{2}+k+d_{l'})v_{\frac{1}{2}+k,l'} - \\ &(a+k+b_l)(\sum_{l'=1}^{S}((a+\frac{1}{2}+k+1-d_{l'})\mu_{k+1,l}^{l'} - (a+k+1-b_l)\mu_{k,l}^{l'}))v_{\frac{1}{2}+k,l'} \\ &= -\sum_{l'=1}^{S}\mu_{k,l}^{l'}v_{\frac{1}{2}+k,l'}.\end{aligned}$$

Thus

$$\begin{aligned}&(a+\frac{1}{2}+k-d_{l'})(a-\frac{1}{2}+k+d_{l'})\mu_{k,l}^{l'} - (a+k-b_l)(a-\frac{1}{2}+k+d_{l'})\mu_{k-1,l}^{l'} \\ &-(a+k+b_l)(a+\frac{1}{2}+k+1-d_{l'})\mu_{k+1,l}^{l'} + (a+k+b_l)(a+k+1-b_l)\mu_{k,l}^{l'} \\ &= -\mu_{k,l}^{l'}\end{aligned} \tag{3.7}$$

By (3.1), (3.2), (3.4) and the identity $[L'_1, Y_{\frac{1}{2}}] = 0$ we get that

$$(a+\frac{1}{2}+k+d_{l'})\lambda_{k+1,l'}^{l} = (a+k+1+b_l)\lambda_{k,l'}^{l}, \tag{3.8}$$

By (3.1)-(3.4) and the identity $[L'_{-1}, Y_{\frac{1}{2}}] = Y_{-\frac{1}{2}}$ we have

$$Y_{-\frac{1}{2}}v_{\frac{1}{2}+k,l'} = \sum_{l=1}^{R}((a+k+1-b_l)\lambda_{k,l'}^{l} - (\frac{1}{2}+a+k-d_{l'})\lambda_{k-1,l'}^{l})v_{k,l}. \tag{3.9}$$

Then by (3.1), (3.2), (3.4), (3.9) and identity $[L'_1, Y_{-\frac{1}{2}}] = -Y_{\frac{1}{2}}$, we have

$$\begin{aligned}&(a+k+b_l)(a+k+1-b_l)\lambda_{k,l'}^{l} - (a+k+b_l)(a+\frac{1}{2}+k-d_{l'})\lambda_{k-1,l'}^{l} - \\ &(a+k+\frac{1}{2}+d_{l'})(a+k+2-b_l)\lambda_{k+1,l'}^{l} + (a+k+\frac{1}{2}+d_{l'})(a+\frac{3}{2}+k-d_{l'})\lambda_{k,l'}^{l} \\ &= -\lambda_{k,l'}^{l}.\end{aligned} \tag{3.10}$$

If $a-\frac{1}{2}+k+d_{l'}\neq 0$ for any $k\in\mathbb{Z}$, by multiplying both sides of (3.7) by $a-\frac{1}{2}+k+d_{l'}$ and then using (3.5), we obtain that

$$(-4k^2+\xi k+\varsigma)\mu_{k,l}^{l'}=-(a-\frac{1}{2}+k+d_{l'})\mu_{k,l}^{l'}$$

for any $k\in\mathbb{Z}$, where $\xi,\varsigma\in\mathbb{C}$. Thus there exists at most two integers, say $k_1,k_2$, such that for any $k\in\mathbb{Z}\setminus\{k_1,k_2\}$, $\mu_{k,l}^{l'}=0$ holds. By (3.5), $\mu_{k_1,l}^{l'}=0=\mu_{k_2,l}^{l'}$. Then

$$\mu_{k,l}^{l'}=0,\forall k,l,l'.$$

Thus by (3.3), we have

$$Y_{\frac{1}{2}}v_{k,l}=0.$$

By a similar discussion on (3.2), (3.8) – (3.10), we get that

$$Y_{\frac{1}{2}}v_{\frac{1}{2}+k,l'}=0.$$

If there exists $k\in\mathbb{Z}$ such that $a-\frac{1}{2}+k+d_{l'}=0$, then we have

$$a=0,d_{l'}=\frac{1}{2}$$

since $0\leq a<\frac{1}{2}, d_{l'}\in\frac{1}{2}+\sqrt{-1}\mathbb{R}$. Then by (3.5) and (3.8) we have

$$\mu_{k,l}^{l'}=0,\forall k\geq 0. \tag{3.11}$$

$$\lambda_{k,l'}^{l}=0,\forall k\leq 0. \tag{3.12}$$

For $k=-1$, note that $b_l\in\frac{1}{2}+\sqrt{-1}\mathbb{R}$, we have

$$Y_{\frac{1}{2}}v_{-1,l}=-\frac{1}{b_l}Y_{\frac{1}{2}}L_{-1}^{'}v_{0,l}=\frac{1}{b_l}Y_{-\frac{1}{2}}v_{0,l}-\frac{1}{b_l}L_{-1}Y_{\frac{1}{2}}v_{0,l}=\frac{1}{b_l}Y_{-\frac{1}{2}}v_{0,l},$$

Then by (3.3) and (3.6), we have

$$\mu_{-1,l}^{l'}=0.$$

Note that $k-1+b_l,-\frac{1}{2}+k+d_{l'}\neq 0$ for all $k\leq -1$, so by (3.5) we have

$$\mu_{k,l}^{l'}=0,\forall k<0. \tag{3.13}$$

Since

$$Y_{\frac{1}{2}}v_{\frac{1}{2}+1,l'}=Y_{\frac{1}{2}}L_1^{'}v_{\frac{1}{2},l'}=L_1^{'}Y_{\frac{1}{2}}v_{\frac{1}{2},l'}=0,$$

we have $\lambda_{1,l'}^{l}=0$, then by (3.8) we have

$$\lambda_{k,l'}^{l}=0,\forall k>0. \tag{3.14}$$

From (3.11)-(3.14) we get that $Y_{\frac{1}{2}}v_{k,l} = 0 = Y_{\frac{1}{2}}v_{\frac{1}{2}+k,l'}$, $\forall k, l$, as required.

**Subcase 2.2.** $W \neq 0$.

Choose an arbitrary nonzero element $w \in W$. $V$ generated by $w$ since $V$ is an irreducible $\mathfrak{sv}$-module. If $Y_{\frac{1}{2}}w = 0$, then $Y.W = M.W = 0$ and $V$ is a trivial $\mathfrak{sv}$-module, a contradiction. Thus $Y_{\frac{1}{2}}w \neq 0$. By $L_0'Y_{\frac{1}{2}}w = \frac{1}{2}Y_{\frac{1}{2}}w$, we see that

$$Y_{\frac{1}{2}}w \in A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K}. \tag{3.15}$$

Moreover,

$$A_{a_i,b_i} \simeq A_{0,b_i} \text{ or } A_{\frac{1}{2},b_i}$$

for each $i \in \{1, \cdots, K\}$ since $V$ is generated by the eigenvector $w$ of $L_0'$ with eigenvalue 0. So

$$V = A_{0,b_1} \oplus \cdots \oplus A_{0,b_R} \oplus A_{\frac{1}{2},d_1} \cdots \oplus A_{\frac{1}{2},d_S} \oplus W.$$

Choose the standard basis $\{v_{k,i} \mid k \in \mathbb{Z}\}$ and $\{v_{\frac{1}{2}+k,j} \mid k \in \mathbb{Z}\}$ for each $A_{0,b_i}$ and $A_{\frac{1}{2},d_j}$ respectively. Suppose

$$Y_{\frac{1}{2}}v_{k,l} = \sum_{l'=1}^{S} \mu_{k,l}^{l'} v_{\frac{1}{2}+k,l'} + w_{k,l},$$

$$Y_{\frac{1}{2}}v_{\frac{1}{2}+k,l'} = \sum_{l=1}^{R} \lambda_{k,l'}^{l} v_{k+1,l} + w_{k,l'}.$$

where $w_{k,l}, w_{k,l'} \in W$. By a similar calculation as that from identity (3.5) to identity (3.14) in Subcase 2.1 we have

$$w_{k,l} = 0 = Y_{\frac{1}{2}}v_{k,l}, \tag{3.16}$$

$$w_{k,l'} = 0 = Y_{\frac{1}{2}}v_{\frac{1}{2}+k,l'}(k \neq 0), \tag{3.17}$$

$$w_{0,l'} = Y_{\frac{1}{2}}v_{\frac{1}{2},l'}, \tag{3.18}$$

and

$$Y_{-\frac{1}{2}}v_{\frac{1}{2}+k,l'} = (k+1-d_{l'})w_{k-1,l'}. \tag{3.19}$$

For any $m \in \mathbb{Z}$, $L_m'Y_{\frac{1}{2}}v_{\frac{1}{2},l'} = L_m'w_{0,l'} = 0$, so

$$Y_{\frac{1}{2}+m}v_{\frac{1}{2},l'} = Y_{\frac{1}{2}}v_{\frac{1}{2}+m,l'} = 0$$

for $m \neq 0, 1$. Thus

$$w_{0,l'} = Y_{\frac{1}{2}}v_{\frac{1}{2},l'} = -[L_1, Y_{-\frac{1}{2}}]v_{\frac{1}{2},l'} = -L_1Y_{-\frac{1}{2}}v_{\frac{1}{2},l'} + Y_{-\frac{1}{2}}L_1v_{\frac{1}{2},l'} = (1+d_{l'})(2-d_{l'})w_{0,l'}.$$

Note that $d_{l'} \in \frac{1}{2}+\sqrt{-1}\mathbb{R}$, we have

$$w_{0,l'} = 0. \tag{3.20}$$

By (3.16)-(3.20), we have

$$Y.(A_{0,b_1} \oplus \cdots \oplus A_{0,b_R} \oplus A_{\frac{1}{2},d_1} \cdots \oplus A_{\frac{1}{2},d_S}) = 0, \tag{3.21}$$

and

$$M.(A_{0,b_1} \oplus \cdots \oplus A_{0,b_R} \oplus A_{\frac{1}{2},d_1} \cdots \oplus A_{\frac{1}{2},d_S}) = 0. \tag{3.22}$$

By (3.15),(3.21) and (3.22), we see that

$$Y(Y_{\frac{1}{2}}w) = M(Y_{\frac{1}{2}}w) = 0. \tag{3.23}$$

Note that $Y_{\frac{1}{2}}w \neq 0$ is also a generator of $V$, combining with (3.23) and (3.15), we have

$$w \in U(L)(Y_{\frac{1}{2}}w) \subseteq A_{a_1,b_1} \oplus \cdots \oplus A_{a_K,b_K},$$

this contradicts with that $0 \neq w \in W$. Thus Subcase 2.2 is impossible. This completes the proof of Proposition 3.4. $\Box$

If we denote the unitary weight modules over the Schrödinger-Virasoro algebra $\mathfrak{sv}$ by $\overline{V}_{\lambda,0,0}, \underline{V}_{\lambda,0,0}$ and $A_{a,b,0,0}$ corresponding respectively to the irreducible unitary highest weight $Vir$-module $\overline{V}_\lambda$, the irreducible unitary lowest weight $Vir$-module $\underline{V}_\lambda$ and the irreducible unitary $Vir$-module $A_{a,b}$. Then Proposition 3.4 and Lemma 3.3 give the classification of the irreducible unitary weight modules over $\mathfrak{sv}$:

**Theorem 3.5.** An irreducible unitary weight module $V$ over the Schrödinger-Virasoro algebra is the highest weight module $\overline{V}_{\lambda,0,0}$, or lowest weight module $\underline{V}_{\lambda,0,0}$ for some $\lambda \in \mathfrak{h}$, or $V$ is isomorphic to $A_{a,b,0,0}$ for some $a \in \mathbb{R}, b \in \frac{1}{2}+\sqrt{-1}\mathbb{R}$.

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# 4 Note

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